12-17-2018

# On the absence of a normal nonabelian Sylow subgroup

Mark W. Bisslier
*Western Governors University*

Jacob Laubacher
*St. Norbert College*

Corey F. Lyons
*Colorado State University*

# ON THE ABSENCE OF A NORMAL NONABELIAN SYLOW SUBGROUP

MARK W. BISSLER, JACOB LAUBACHER, AND COREY F. LYONS

Abstract. Let $G$ be a finite solvable group. We show that $G$ does not have a normal nonabelian Sylow $p$-subgroup when its prime character degree graph $\Delta(G)$ satisfies a technical hypothesis.

## 1. Introduction and Preliminaries

In this paper we fix $G$ to be a finite solvable group. Set $\mathbf{F}(G)$ to be the Fitting subgroup of $G$, write $\mathrm{Irr}(G)$ for the set of irreducible characters of $G$, and let $\mathrm{cd}(G)$ denote the set of irreducible character degrees of $G$. Denote $\rho(G)$ to be the set of primes that divide degrees in $\mathrm{cd}(G)$. The prime character degree graph of $G$, written $\Delta(G)$, is the graph whose vertex set is $\rho(G)$. Two vertices $p$ and $q$ of $\Delta(G)$ are adjacent if there exists $d \in \mathrm{cd}(G)$ such that $pq$ divides $d$. In this paper, we concern ourselves solely with graphs $\Delta(G)$ that are connected with $\rho(G) \geq 5$.

We fix the following notation for an arbitrary vertex $p \in \rho(G)$:

$$\pi := \{q \in \rho(G) \ : \ q \text{ is adjacent to } p \text{ in } \Delta(G)\}$$

and

$$\rho := \{q \in \rho(G) \ : \ q \text{ is not adjacent to } p \text{ in } \Delta(G)\}.$$

Observe that $\rho$ induces a complete subgraph in $\Delta(G)$ due to Pálfy's condition from [8], and that $\rho(G)$ is a disjoint union: $\{p\} \cup \pi \cup \rho$. We let $\pi^*$ and $\rho^*$ denote arbitrary nonempty subsets of $\pi$ and $\rho$, respectively. Finally, let $\pi^* \cup \rho^*$ be an arbitrary vertex set which induces a complete subgraph in $\Delta(G)$. Let $\beta$ be a subset of $\rho(G)$ that contains $\pi^* \cup \rho^*$ such that $\beta$ also induces a complete subgraph in $\Delta(G)$. Set $\mathcal{B}$ to be the union of all such $\beta$'s satisfying these properties for $\pi^* \cup \rho^*$. Consider $\tau := \mathcal{B} \setminus (\pi^* \cup \rho^*)$, and denote $\tau^*$ to be a subset of $\tau$. Notice that $\tau$ could be empty, depending on the initial set $\pi^* \cup \rho^*$.

In this note we build upon the work done in [2] and [6] which concerns a normal nonabelian Sylow $p$-subgroup of a finite solvable group. We use tools from [2] which classify vertices as admissible. Recall the definition and result directly related to admissible vertices below.

**Definition 1.1.** ([2]) A vertex $q$ of a graph $\Gamma$ is **admissible** if:

(i) the subgraph of $\Gamma$ obtained by removing $q$ and all edges incident to $q$ does not occur as the prime character degree graph of any solvable group, and
(ii) none of the subgraphs of $\Gamma$ obtained by removing one or more of the edges incident to $q$ occur as the prime character degree graph of any solvable group.

*Date*: November 9, 2018.
2010 *Mathematics Subject Classification.* Primary 20D10; Secondary 20D20.
*Key words and phrases.* finite solvable groups, normal nonabelian subgroup, character degree graphs.

**Lemma 1.2** (Lemma 2.1 from [2])**.** *Let $q$ be an admissible vertex of $\Delta(G)$. For every proper normal subgroup $N$ of $G$, suppose that $\Delta(N)$ is a proper subgraph of $\Delta(G)$. Then $O^q(G) = G$.*

Most notably, our results pertain to groups $G$ and graphs $\Delta(G)$ which do not satisfy Lemmas 2.3 and 2.4 from [2]. They also conclude in those lemmas that there is no normal nonabelian Sylow $p$-subgroup. This is a typical approach en route to showing that a graph cannot occur as the prime character degree graph of any solvable group.

Next we recall several lemmas due to Lewis which pertain to groups having a normal nonabelian Sylow subgroup.

**Lemma 1.3** (Lemma 4.2 from [6])**.** *Let $G$ be a solvable group with a normal nonabelian Sylow $p$-subgroup $P$. Let $H$ be a $p$-complement for $G$. Assume that $H$ acts faithfully on $P$, that $\rho \subseteq \rho(H)$, and that the Fitting subgroup of $H$ is a $\pi$-group. Then $\rho(H) = \rho$.*

**Lemma 1.4** (Lemma 4.3 from [6])**.** *Let $G$ be a solvable group with a normal nonabelian Sylow $p$-subgroup $P$. Let $H$ be a $p$-complement for $G$. Suppose that $P'$ is minimal normal in $G$, and $\rho(H) = \rho$ is not empty. Then there is a nonempty subset $\pi^*$ of $\pi$ so that $\Delta(P'H)$ has two connected components: $\rho$ and $\pi^*$. Furthermore, if $|\rho| = n$, then $|\pi^*| \geq 2^n - 1$.*

**Lemma 1.5** (Lemma 3 from [5])**.** *Let $G$ be a solvable group and let $p \in \rho(G)$. If $P$ is a normal Sylow $p$-subgroup, then $\rho(G/P') = \rho(G) \setminus \{p\}$.*

To conclude this section, we recall two powerful results concerning when the prime character degree graph is disconnected.

**Theorem 1.6** (Theorem 5.5 from [4])**.** *Let $G$ be a solvable group and suppose that $\Delta(G)$ has two connected components. Then there is precisely one prime $p$ so that the Sylow $p$-subgroup of the Fitting subgroup of $G$ is not central in $G$.*

**Theorem 1.7** (Pálfy's inequality from [9])**.** *Let $G$ be a solvable group and $\Delta(G)$ its prime character degree graph. Suppose that $\Delta(G)$ is disconnected with two components having size $a$ and $b$, where $a \leq b$. Then $b \geq 2^a - 1$.*

## 2. Main Results

In this section we prove that $G$ has no normal nonabelian Sylow $p$-subgroup. Hypothesis 2.1 below is strictly about the prime character degree graph $\Delta(G)$, whereas Hypothesis 2.2 is about the group $G$ itself. Further, we always assume $|G|$ is minimal.

**Hypothesis 2.1.** Concerning $\Delta(G)$, we assume the following:

(i) for every vertex in $\rho$, there exists a nonadjacent vertex in $\pi$,
(ii) for every vertex in $\pi$, there exists a nonadjacent vertex in $\rho$,
(iii) all the vertices in $\pi$ are admissible. Moreover, no proper connected subgraph with vertex set $\{p\} \cup \pi^* \cup \rho$ occurs as the prime character degree graph of any solvable group,
(iv) for each vertex set $\pi^* \cup \rho^*$ which induces a complete subgraph in $\Delta(G)$, all the vertices in the corresponding set $\tau$ are admissible. Moreover, no proper connected subgraph with vertex set $\rho(G) \setminus \tau^*$ occurs as the prime character degree graph of any solvable group, and
(v) if a disconnected subgraph with vertex set $\rho(G)$ does not occur, then it must specifically violate Pálfy's inequality from [9]. Finally, if a disconnected subgraph with vertex set $\rho(G)$ does occur, then the sizes of the connected components must be $n > 1$ and $2^n - 1$.

Observe the following consequences of Hypothesis 2.1. For (i) and (ii), since $\Delta(G)$ is assumed to be connected, it is required that $|\pi| \geq 2$ and $|\rho| \geq 2$. As another consequence of (ii), all vertices in $\pi$ are adjacent to each other. Otherwise, if there existed two nonadjacent vertices in $\pi$, then the complement graph would have an odd cycle. This is prohibited by the main theorem from [1]. Concerning (iii), one gleans that no proper connected subgraph with vertex set $\rho(G)$ occurs as the prime character degree graph of any solvable group (taking $\pi^* = \pi$). Depending on the initial subset $\pi^* \cup \rho^*$, (iv) may require that some of the vertices in $\rho$ are admissible. Finally, we know from (v) that if a disconnected subgraph with vertex set $\rho(G)$ does indeed occur as the prime character degree graph of a solvable group, then it must be represented by Example 2.4 from [6].

**Hypothesis 2.2.** Concerning the group $G$, we assume $P$ is a normal nonabelian Sylow $p$-subgroup of $G$, and we let $H$ be a $p$-complement for $G$. We set $F := \mathbf{F}(H)$, which is necessarily nontrivial.

Concerning $F$, it is known there exists a degree in $\mathrm{cd}(G)$ that is divisible by all the prime divisors of $|F|$. This corresponds to the vertices in $\pi(|F|)$ inducing a complete subgraph in $\Delta(G)$. Next, assuming $G$ has a normal nonabelian Sylow $p$-subgroup, we can now see the full force of Hypothesis 2.1(v). As stated above, this hypothesis forces the disconnected graph (should it occur) to be represented by Example 2.4 from [4]. In that case, all normal Sylow subgroups are necessarily abelian. Therefore, assuming $N$ is a group such that $\rho(N) = \rho(G)$, we in fact get $\Delta(N) = \Delta(G)$ by Hypothesis 2.1(iii). Since $|G|$ is minimal we conclude that $N = G$.

**Lemma 2.3.** *Assume Hypotheses 2.1 and 2.2. Then* $C_H(P) = 1$.

*Proof.* Set $C := C_H(P)$. We follow the argument on page 259 in [6] in a generalized form to show $C = 1$.

First we observe that $C$ is normal in $G$, and that $PC = P \times C$. Therefore, any prime in $\rho(C)$ must be adjacent to $p$ in $\Delta(G)$. So we know that $\rho(C) \subseteq \pi$ and that $r$ divides $|H : C|$ for every $r \in \rho$. Since we know that $H/C$ acts faithfully on $P$, we see that the primes in $\pi(|H : C|)$ must lie in $\rho(G/C)$. Thus, $\rho \subseteq \rho(G/C)$. We then conclude that $\rho(G/C)$ must have one of the following vertex sets: (a) $\{p\} \cup \rho$, (b) $\{p\} \cup \pi^* \cup \rho$, or (c) $\{p\} \cup \pi \cup \rho$. Here $\pi^*$ is taken as a nonempty proper subset of $\pi$.

Suppose (a). Note that none of the primes in $\pi$ divide $|H : C|$. We fix a nonlinear character $\gamma \in \mathrm{Irr}(P)$ and then $\gamma$ extends to $\gamma \times 1_C$ in $P \times C$. For $\chi \in \mathrm{Irr}(G|\gamma \times 1_C)$ notice that $p$ divides $\chi(1)$ and $\chi(1)/\gamma(1)$ divides $|H : C|$. Since none of the primes in $\rho$ divide $\chi(1)$ and the only primes that could possibly divide $|H : C|$ are the primes in $\rho$, we are forced to conclude that $\chi_{PC} = \gamma \times 1_C$, which implies $\chi_P = \gamma$. Using Gallagher's Theorem (Corollary 6.17 from [3]) we have that $\chi(1)d \in \mathrm{Irr}(G)$ for every degree $d \in \mathrm{cd}(H)$. Thus, $\rho(H) \subseteq \pi$. This means that $H$ has a normal abelian Hall $\rho$-subgroup, and hence $O^q(G) < G$ for some $q \in \pi$. This is a contradiction since every vertex in $\pi$ is admissible.

Suppose (b). By Hypothesis 2.1(iii), we know that any connected subgraph with vertex set $\{p\} \cup \pi^* \cup \rho$ does not occur as the prime character degree graph of any solvable group. If $\Delta(G/C)$ is a disconnected graph that does occur, then by Theorem 5.5 from [4] we conclude that $G/C$ has a central Sylow $q$-subgroup for some $q \in \pi \setminus \pi^*$. This implies $O^q(G) < G$, which is a contradiction since every vertex in $\pi$ is admissible.

We are left with (c), in which case $\rho(G/C) = \{p\} \cup \pi \cup \rho = \rho(G)$. By Hypothesis 2.1(v), we see that $|G/C| = |G|$, and hence $C = 1$. □

Notice that Lemma 2.3 implies that $H$ acts faithfully on $P$. With this in hand, we now investigate $F$, the Fitting subgroup of $H \cong G/P$.

**Claim 2.4.** *Assume Hypotheses 2.1 and 2.2. Then $F$ is not a $\rho$-subgroup.*

*Proof.* For the sake of contradiction, we assume that $F$ is a $\rho$-subgroup. Let $\gamma \in \mathrm{Irr}(P)$ be a nonlinear character and let $\chi \in \mathrm{Irr}(G|\gamma)$. We see that $p$ divides both $\gamma(1)$ and $\chi(1)$, and that $r$ does not divide $\chi(1)$ or $\chi(1)/\gamma(1)$ for all $r \in \rho$. Since this is true for all characters $\chi \in \mathrm{Irr}(G|\gamma)$, we apply Theorem 12.9 of [7] to get that $G/P \cong H$ has an abelian Hall $\rho$-subgroup.

Since we assumed that $F$ was in fact a $\rho$-subgroup, we now get that $F$ is the abelian Hall $\rho$-subgroup. Thus $O^q(G) < G$ for some $q \in \pi$, which is a contradiction since all the vertices in $\pi$ are admissible. Therefore $F$ is not a $\rho$-subgroup. □

**Claim 2.5.** *Assume Hypotheses 2.1 and 2.2. Then $F$ is not a $\pi^* \cup \rho^*$-subgroup.*

*Proof.* For the sake of contradiction, we assume that $F$ is a $\pi^* \cup \rho^*$-subgroup. It is sufficient to suppose $\pi(|F|) = \pi^* \cup \rho^*$. Observe that in this scenario, Hypothesis 2.1(i) and (ii) force $\pi^*$ and $\rho^*$ to be proper subsets of $\pi$ and $\rho$, respectively.

We fix the following notation:

$$\eta := \{s \in \rho(G) \setminus (\{p\} \cup \pi(|F|)) \ : \ s \text{ and } q \text{ are not adjacent in } \Delta(G) \text{ for some } q \in \pi(|F|)\}.$$

In fact, $\eta = \rho(G) \setminus (\{p\} \cup \mathcal{B})$ and $\eta$ must contain at least one element from $\pi \setminus \pi^*$ and from $\rho \setminus \rho^*$. For otherwise this would contradict Hypothesis 2.1(i) and (ii) as $\pi^* \cup \rho^*$ forms a complete subgraph of $\Delta(G)$. Thus $\eta$ is nonempty.

Let $\gamma \in \mathrm{Irr}(PF)$ be such that $q$ divides $\gamma(1)$ for all $q \in \pi^* \cup \rho^*$. For any character $\chi \in \mathrm{Irr}(G|\gamma)$, we have that $q$ divides $\chi(1)$ for all $q \in \pi^* \cup \rho^*$, and that $r$ does not divide $\chi(1)$ or $\chi(1)/\gamma(1)$ for any $r \in \eta$. By Theorem 12.9 from [7], we see that $G/PF \cong H/F$ has an abelian Hall $\eta$-subgroup. Let $M = O_{\mathcal{B}}(H)$, and note that $F \subseteq M$. Set $E/M$ to be the Fitting subgroup of $H/M$. Since $H/M$ has an abelian Hall $\eta$-subgroup, we see that $E/M$ is the abelian Hall $\eta$-subgroup.

Next we have that $\rho(PE)$ contains $\{p\} \cup \pi^* \cup \rho^* \cup \eta$. However this may be a proper subset of $\rho(G)$. Notice that by construction we have that $\rho(G) \setminus (\{p\} \cup \pi^* \cup \rho^* \cup \eta) = \tau$. Hence $\rho(PE)$ must be one of the following: (a) $\rho(G) \setminus \tau^*$ or (b) $\rho(G)$.

Suppose (a). Then by Hypothesis 2.1(iv) we see that no proper connected subgraph with vertex set $\rho(G) \setminus \tau^*$ occurs as the prime character degree graph of any solvable group. If $\Delta(PE)$ is a disconnected graph that occurs, then by Theorem 5.5 from [4] we observe that $PE$ has a central Sylow $q$-subgroup for some $q \in \tau^*$. This implies $O^q(G) < G$, which is a contradiction since every vertex in $\tau$ is admissible.

We are left with (b). So $\rho(PE) = \rho(G)$, and by Hypothesis 2.1(v) we see that $PE = G$. Since $E/M$ is abelian, and since $\eta \cap \pi \neq \varnothing$, we have that $O^s(G) < G$ for some $s \in \pi$. This is a contradiction since all the vertices in $\pi$ are admissible. Therefore $F$ is not a $\pi^* \cup \rho^*$-subgroup. □

**Claim 2.6.** *Assume Hypotheses 2.1 and 2.2. Then $F$ is not a $\pi$-subgroup.*

*Proof.* For the sake of contradiction, we suppose $F$ is a $\pi$-subgroup of $H$ and hence $\rho \subseteq \rho(H)$. Furthermore, $H$ acts faithfully on $P$ as consequence of Lemma 2.3. We apply Lemma 4.2 from [6] which yields $\rho(H) = \rho$.

Next we will see that $P'$ is minimal normal in $G$. Suppose that $X$ is normal in $G$ such that $1 \leq X \leq P'$ and $P'/X$ is a chief factor for $G$. Our goal will be to show that $X = 1$.

We know $\rho(G/X) \supseteq \rho(G/P') = \rho(G) \setminus \{p\}$ (by Theorem 3 from [5]). However, $p \in \rho(G/X)$ since $G/X$ has a normal nonabelian Sylow $p$-subgroup. It follows that $\rho(G) = \rho(G/X)$, and hence $|G/X| = |G|$ by Hypothesis 2.1(v). Therefore $X = 1$ and $P'$ is minimal normal in $G$.

The hypotheses of Lemma 4.3 from [6] are satisfied, and therefore $\Delta(P'H)$ has disconnected components with vertex sets $\rho$ and $\pi^*$ such that

$$|\pi^*| \geq 2^{|\rho|} - 1. \tag{1}$$

In particular, notice that

$$|\rho| \leq |\pi^*|. \tag{2}$$

Hypothesis 2.1(v) gives two cases concerning the possibility of a disconnected subgraph of $\Delta(G)$ with vertex set $\rho(G)$. We will investigate these below, but first notice that by our assumptions, the two connected components of a possibly disconnected subgraph of $\Delta(G)$ with vertex set $\rho(G)$ are forced to have vertex sets $\rho$ and $\{p\} \cup \pi$.

For Case 1, we consider if no disconnected subgraph of $\Delta(G)$ with vertex set $\rho(G)$ occurs. By Hypothesis 2.1(v) we know that for component sizes $a$ and $b$ (with $a \leq b$), we must have the situation where $b < 2^a - 1$. This gives two subcases:

Case 1(a): $|\rho| = a$ and $|\{p\} \cup \pi| = b$. For this scenario we have

$$|\pi^*| \leq |\pi| < |\{p\} \cup \pi| = b < 2^a - 1 = 2^{|\rho|} - 1,$$

which contradicts (1).

Case 1(b): $|\{p\} \cup \pi| = a$ and $|\rho| = b$. Thus

$$|\rho| = b \geq a = |\{p\} \cup \pi| > |\pi| \geq |\pi^*|,$$

which contradicts (2).

For Case 2, we consider if a disconnected subgraph of $\Delta(G)$ with vertex set $\rho(G)$ occurs. By Hypothesis 2.1(v) we know the components must be of sizes $n > 1$ and $2^n - 1$, which gives two subcases:

Case 2(a): $|\rho| = n > 1$ and $|\{p\} \cup \pi| = 2^n - 1$. Observe that

$$|\pi^*| \leq |\pi| < |\{p\} \cup \pi| = 2^n - 1 = 2^{|\rho|} - 1,$$

which contradicts (1).

Case 2(b): $|\{p\} \cup \pi| = n > 1$ and $|\rho| = 2^n - 1$. Similarly notice

$$|\rho| = 2^n - 1 > n = |\{p\} \cup \pi| > |\pi| \geq |\pi^*|,$$

which contradicts (2). Therefore $F$ is not a $\pi$-subgroup. □

We have shown that no primes in $\rho(G) \setminus \{p\}$ will divide $|F|$. This leads to the following result:

**Theorem 2.7.** *Assume Hypothesis 2.1. Then $G$ has no normal nonabelian Sylow $p$-subgroup.*

*Proof.* For the sake of contradiction, suppose Hypothesis 2.2. That is, suppose $G$ has a normal nonabelian Sylow $p$-subgroup $P$, and let $H$ be a $p$-complement for $G$ with Fitting subgroup $F$.

By Claims 2.4, 2.5, and 2.6, we see that $F$ must be trivial since no primes divide $|F|$. As $H$ is solvable it must have a nontrivial Fitting subgroup, a contradiction. □

Consider the following graphs with the appropriate vertex $p$ labeled in Figure 1 for several examples and applications of Theorem 2.7.

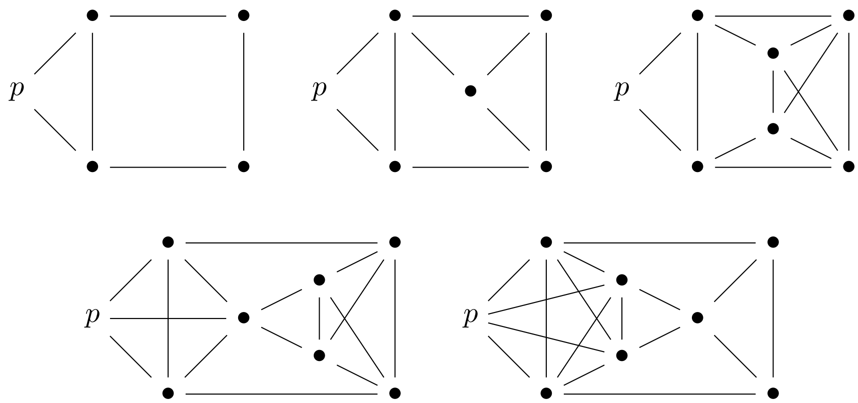


FIGURE 1. Graphs satisfying Hypothesis 2.1

## REFERENCES

[1] Zeinab Akhlaghi et al. On the character degree graph of solvable groups. *Proc. Amer. Math. Soc.*, 146(4):1505–1513, 2018.
[2] Mark W. Bissler and Mark L. Lewis. A family of graphs that cannot occur as character degree graphs of solvable groups. arXiv:1707.03020, 2017.
[3] I. Martin Isaacs. *Character theory of finite groups.* Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)].
[4] Mark L. Lewis. Solvable groups whose degree graphs have two connected components. *J. Group Theory*, 4(3):255–275, 2001.
[5] Mark L. Lewis. Solvable groups with character degree graphs having 5 vertices and diameter 3. *Comm. Algebra*, 30(11):5485–5503, 2002.
[6] Mark L. Lewis. Classifying character degree graphs with 5 vertices. In *Finite groups 2003*, pages 247–265. Walter de Gruyter, Berlin, 2004.
[7] Olaf Manz and Thomas R. Wolf. *Representations of solvable groups.* Cambridge University Press: Cambridge, 1993.
[8] Péter Pál Pálfy. On the character degree graph of solvable groups. I. Three primes. *Period. Math. Hungar.*, 36(1):61–65, 1998.
[9] Péter Pál Pálfy. On the character degree graph of solvable groups. II. Disconnected graphs. *Studia Sci. Math. Hungar.*, 38:339–355, 2001.

DEPARTMENT OF GENERAL EDUCATION - COLLEGE OF BUSINESS, WESTERN GOVERNORS UNIVERSITY, SALT LAKE CITY, UTAH 84107
*E-mail address*: mark.bissler@wgu.edu

DEPARTMENT OF MATHEMATICS, ST. NORBERT COLLEGE, DE PERE, WISCONSIN 54115
*E-mail address*: jacob.laubacher@snc.edu

DEPARTMENT OF MATHEMATICS AND PHYSICS, COLORADO STATE UNIVERSITY - PUEBLO, PUEBLO, COLORADO 81001
*E-mail address*: coreylyons314@gmail.com