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# Classifying Prime Character Degree Graphs

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# Classifying Prime Character Degree Graphs

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## Introduction

We operate under the following convention:

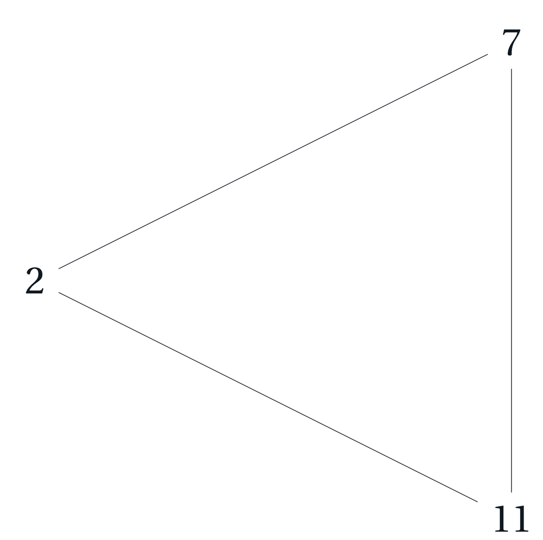
- $G$ is a finite solvable group
- $\mathrm{Irr}(G)$ denotes the set of irreducible characters of $G$
- $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$
- $\Delta(G)$ is the prime character degree graph of $G$
- $\rho(G)$ denotes the vertices of $\Delta(G)$, which consists of all such prime divisors of $\mathrm{cd}(G)$
- There is an edge between two distinct vertices $p, q \in \rho(G)$ if there exists some character $a \in \mathrm{cd}(G)$ such that $pq \mid a$
- $\Delta(G)$ is a simple graph (so no direction on edges, no multiple edges, no loops on a vertex)

**Claim 1.** *Given any $G$, we can find $\Delta(G)$.*

**Example 2.** Let $G$ be such that $\mathrm{cd}(G) = \{1, 6\}$.


**Example 3.** Let $G$ be such that $\mathrm{cd}(G) = \{1, 2, 4, 8, 16, 32, 45\}$.


**Example 4.** Let $G$ be such that $\mathrm{cd}(G) = \{1, 14, 22, 77\}$.




## Goal

As stated in Claim 1, we can always find the prime character degree graph $\Delta(G)$ for any group $G$. This isn't exciting.

In our project, we essentially want to do the opposite. That is, given any graph, we want to decide if there is some group $G$ so that its prime character degree graph $\Delta(G)$ is the same as the graph we started with. Specifically, we will work with a family of graphs constructed in a particular way.

## Construction

Let $1 \le n \le k$. We now construct the family of graphs denoted by $\{\Sigma^*_{k,n}\}$. For subgraphs $A$ and $B$, and a fixed vertex $c$, we have that:

(i) $A$ is a complete graph on $k$ vertices $a_1, a_2, \dots, a_k$,

(ii) $B$ is a complete graph on $k+n$ vertices $b_1, b_2, \dots, b_k, \dots, b_{k+n}$,

(iii) $c \notin \rho(A)$ and $c \notin \rho(B)$,

(iv) $\rho(A) \cap \rho(B) = \varnothing$,

(v) there is an edge between $a_i$ and $b_i$ for all $1 \le i \le k$,

(vi) there is an edge between $a_i$ and $b_{k+i}$ for all $1 \le i \le n$,

(vii) there is an edge between $c$ and $a_i$ for all $1 \le i \le k$ in $\Sigma^L_{k,n}$,

(viii) there is an edge between $c$ and $b_i$ for all $1 \le i \le k+n$ in $\Sigma^R_{k,n}$,

(ix) there are no edges in the graph $\Sigma^*_{k,n}$ other than the edges described in (i)–(viii).

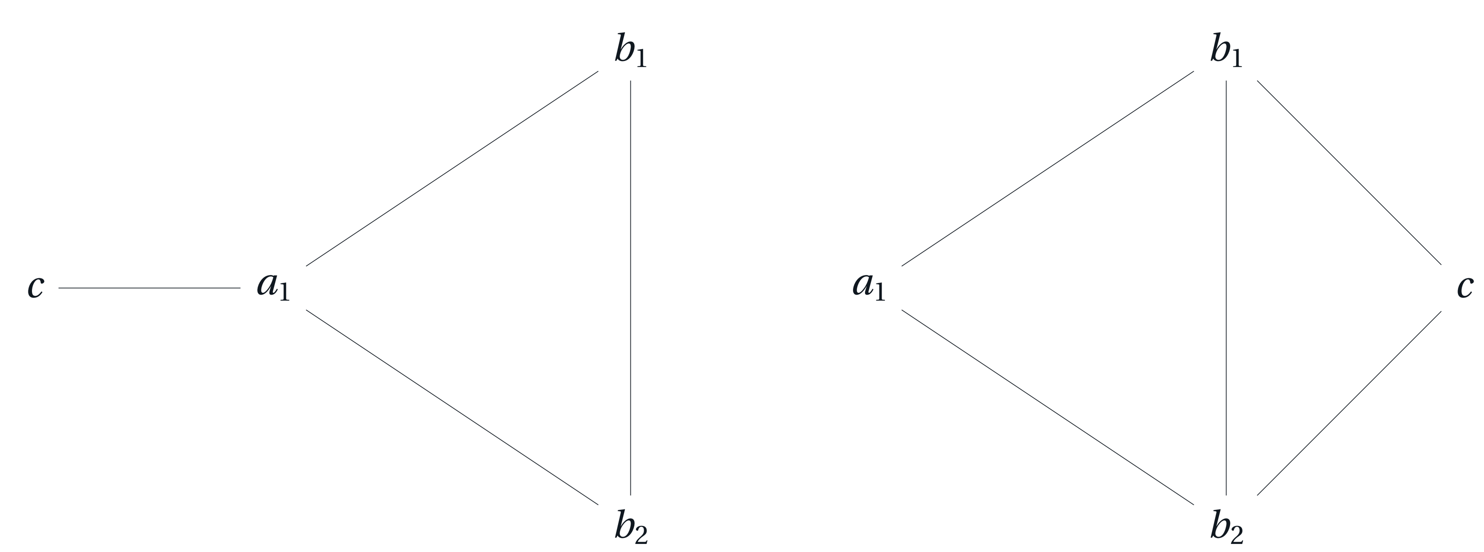
The notation of the graph $\Sigma^*_{k,n}$:

- The graph is read from left to right, with $A$ on the left and $B$ on the right
- There are two variants, $\Sigma^L_{k,n}$ and $\Sigma^R_{k,n}$, where the superscript tracks which side the fixed vertex $c$ resides
- The subscript $k$ represents how many vertices are in $A$
- The subscript $n$ represents how many one-to-two distinct edge mappings exist from $A$ to $B$

## Main Result

Considering the above construction, we have the following:

**Theorem 5.** *The graphs $\Sigma^L_{1,1}$ and $\Sigma^R_{1,1}$ occur as the prime character degree graph of a solvable group (see* Figure 1*). The graphs $\Sigma^R_{2,1}$ and $\Sigma^R_{2,2}$ possibly occur as the prime character degree graph of a solvable group (see* Figure 2 *and* Figure 3*). Otherwise $\Sigma^*_{k,n}$ does not occur as the prime character degree graph of any solvable group.*




The graph $\Sigma^L_{1,1}$ The graph $\Sigma^R_{1,1}$

Figure 1

*Proof.* We proceed by induction on $n$. In particular, we handle the graphs $\Sigma^L_{k,n}$ and $\Sigma^R_{k,n}$ separately, and taking $k \ge 3$, we show that $\Sigma^*_{k,n}$ does not occur. All other cases with a small number of vertices are easily handled separately. The argument heavily relies on knowing information about subgraphs of the graph we're investigating. □

We want to take a closer look at the graphs that are yet-to-be classified. Specifically, the two graphs: $\Sigma^R_{2,1}$ and $\Sigma^R_{2,2}$.

## The First Graph

We first consider the graph $\Sigma^R_{2,1}$.

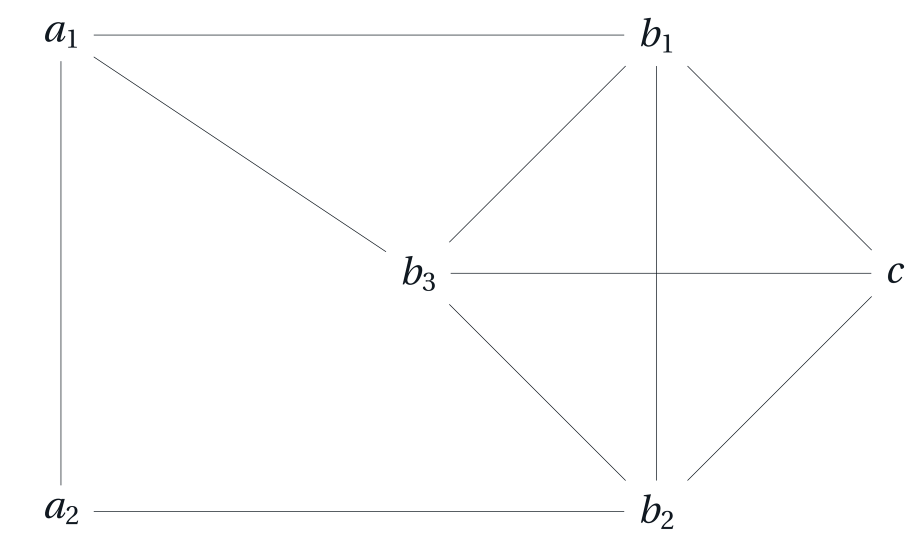



Figure 2

What went wrong:

- Deleting the vertex $a_2$ and all incident edges, along with the edge between $b_1$ and $b_2$, yields a graph with five vertices that is currently unknown as to whether it does or does not occur as the prime character degree graph.

## The Second Graph

Next we consider the graph $\Sigma^R_{2,2}$.

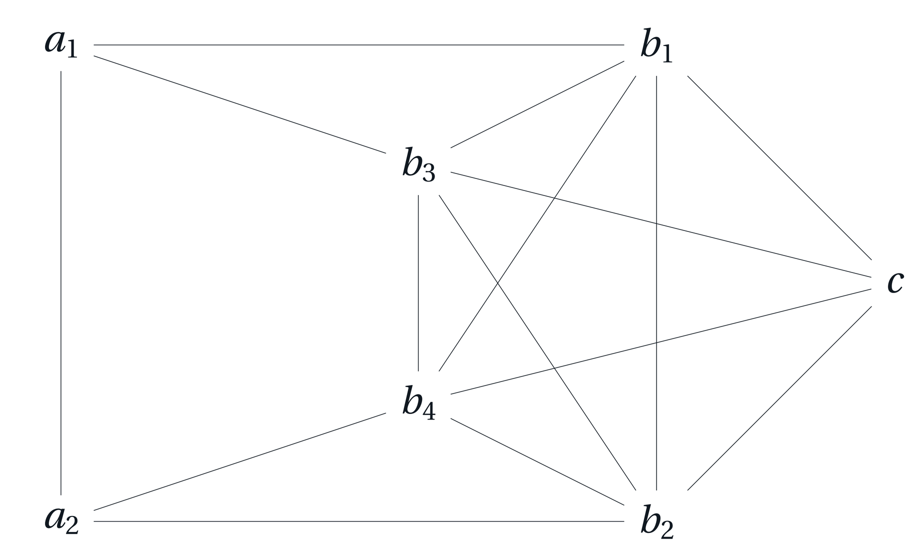



Figure 3

What went wrong:

- The disconnected graph with 2 vertices and 5 vertices occurs, and our strategy used for Theorem 5 required the disconnected graph to not occur.
- Deleting the vertex $b_4$ and all incident edges yields the graph in Figure 2, so we once again don't know information about one of the subgraphs.

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